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The Common Cycle Economic Lot Scheduling Problem with Backorders: Benefits of Controllable Production Rates

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Abstract. In this paper, we deal with the production scheduling of several products that are produced periodically, in a fixed sequence, on a single machine. In the literature, this problem is usually referred to as the Common Cycle Economic Lot Scheduling Problem. We extend the latter to allow the production rates to be controllable at the beginning of as well as during each production run of a product. Also, we assume that unsatisfied demand is completely backordered. The objective is to determine the optimal schedule that satisfies the demand for all the products and that realizes the minimum average setup, inventory holding and backlog cost per unit time. Comparison with previous results (when production rates are fixed) reveals that average costs can be reduced up to 66% by allowing controllable production rates.

Key words: Inventory/production control, deterministic, lot-sizing, single-machine.

1. Introduction

The Economic Lot Scheduling Problem (ELSP) deals with the scheduling of the production of several products on one or more identical machines. Each product has a known constant demand rate and a fixed production rate when it is being produced. When production is switched from one product to the next a sequence-independent constant setup time as well as a fixed setup cost are incurred. The time horizon is infinite, the system is in steady state and no backlog is allowed. The objective is to determine lot sizes that minimize the average setup and inventory holding costs per unit time. In Gallego (1989), the ELSP is extended to allow backlog. In this paper, we consider the case where non-satisfied demand is totally backlogged.

Although, there has been a large amount of research work on the ELSP, an optimal solution approach has not been proposed yet. Rather, good (some times excellent) heuristics have been suggested. A comprehensive review of the ELSP through 1976 is given in Elmaghraby (1978). Recent work on the ELSP includes the work of Boctor (1982), Hsu (1983), Maxwell and Singh (1983), Axsäter (1985), Goyal (1984), Roundy (1985), Dobson (1987), Gallego (1989), Jones and Inman (1989), Lee and Surya (1989), Carreno (1990), and Zipkin (1991). Most of the aforementioned work emphasized on the feasibility of cyclic schedules. A cyclic schedule can be one of two types: Common Cycle schedule or Basic Period schedule. The Common Cycle schedule is also known as the Rotation Cycle schedule, where production is cycled through the products every T units of time. It is well known that this type of schedule is always feasible. Basically, for the Common Cycle schedule, the sequencing problem is eliminated. In the Basic Period schedule, each product's cycle time is an integer multiple of a basic cycle time. Since the production rates are constant, all lots of each product are of equal size. Usually, The Basic Period schedule gives a lower cost than the Common Cycle schedule. The main problem of the Basic Period schedule is its feasibility. In fact, for this kind of approach, Hsu (1983) has shown that even the problem of finding a feasible schedule is an NP-complete problem. To overcome this feasibility problem, Dobson (1987) suggested a new formulation that allows time varying production runs and which includes setup times explicitly in the problem formulation. The main advantage of this approach is that, it always provides a feasible schedule.

Recently, researchers have dropped the assumption of fixed production rates at the machine and have improved the ELSP model using controllable production rates. The production rate of each product can be chosen to be any value less than a maximum rate. Recent work along this new direction includes the work of Buzacott and Ozkarahan (1983) where they studied the case of a two-product system. First, they categorize the products according to their dollar value of usage, then they showed that only the product with the lower dollar value of usage is produced at maximum. Arizono *et al*. (1989) studied the effects of controllable production rates on inventory systems. They showed that a controllable production rate inventory system is more efficient than one with fixed production rates. Silver (1990) studied an m-product system under the Common Cycle schedule assumption. He showed that at most one product slows down its production rate. The optimal production rates and cycle time length were obtained numerically. In all of the aforementioned work, the production rates were decided at the beginning and supposed to be fixed during a product's production run. Moon *et al*. (1991) generalized Silver's model by considering a system with controllable production rates during the production runs of the products. They obtained the optimal production rates numerically and showed that savings almost twice as large as those reported in the literature can be obtained.

In this paper, we study the same problem as in Moon *et al*. (1991). Our approach is different from theirs in several aspects. First, based on a result from Bai and Elhafsi (1996) and Elhafsi and Bai (1996), we show that the optimal production rates can take on only three values, namely zero, demand rate, maximum rate. Second, we formulate the problem allowing backlog, with the case of no backlog as a special case. Third, we derive the optimal solution of the two-product problem analytically. Finally, for the m-product problem, to obtain the optimal solution numerically, we propose an algorithm based on Zoutendijk's algorithm which requires neither using a line search algorithm (which makes it very fast) nor solving a linear programming sub-problem. Hence, it is much simpler to implement.

The paper is organized as follows. In Section 2, we present the notation and state the assumptions of the model. In Section 3, we study the two-product problem. In Section 4, we extend the formulation to the m-product problem and present an algorithm to obtain the optimal solution numerically. We conclude our study with Section 5.

2. Assumptions and Notation

Assumptions

(1) Only one product can be produced at a time.
(2) The demand rate for each product is known and constant.
(3) The production rate of each product is controllable during its production run.
(4) Setup times are known constants and sequence independent.
(5) Backlog is allowed.
(6) The time horizon is infinite and the system is in steady state.
(7) The machine has enough capacity to satisfy the demand for all products.
(8) The Common Cycle schedule policy is used.

In addition, we assume that the costs of changing production rates are negligible and that the production cost per unit is constant over the infinite planning horizon. Variable production rates can be easily implemented by adjusting the loading time of each part or product type on the machine without incurring a new setup time or cost (provided that it is the same type of product the machine has been set up for previously). For instance, if a particular product has an average demand of 2 units per day (8 working hours) and can be produced at a rate of 4 units per day at most, a production at the demand rate corresponds to loading a part on the machine every 4 hours, while a production at the maximum rate corresponds to loading a part every other hour. For such cases, workers responsible for the loading of parts on the machines are paid regardless of the production rate. Hence, the labor cost (which is the only cost that might be affected by production rate changes) allocated to the production of the parts is incurred regardless of the production rates and therefore would not affect the optimal schedule. For more complicated manufacturing systems, a worker may perform several tasks alternatively during a period of time. Detailed treatment of job assignment for workers is beyond the scope of this paper.

Notation

For the i^{th} product ($i = 1, ..., m$)
d_i demand per unit time
p_i maximum production per unit time

δ_i required setup time
k_i required setup cost
h_i inventory holding cost per unit per unit time
b_i backlog cost per unit per unit time
t_i time spent producing at maximum rate
τ_i time spent producing at the demand rate
$\gamma_i = h_i b_i/(h_i + b_i)$ cost factor
$\rho_i = d_i/p_i$ utilization factor of the machine by product *i*
$A_i = \gamma_i/2d_i(1 - \rho_i)$
$T = \sum_{i=1}^{i=m}(t_i + \tau_i + \delta_i)$ length of the common cycle
$\delta = \sum_{i=1}^{i=m} \delta_i$ total setup time during T
$K = \sum_{i=1}^{i=m} k_i$ total setup cost during T
$\rho = \sum_{i=1}^{i=m} \rho_i$ total utilization factor of the machine.

3. The Two-Product Problem

In this section, we formulate the problem for a system with two products only, then we derive the optimal solution analytically. But first, we introduce the following theorem:

THEOREM 1. *The optimal production rate vector* $u^*(t) = (u_1^*(t), u_2^*(t))$ *belongs to the finite set of vectors* $\Omega^* = \{(0,0), (p_1, 0), (d_1, 0), (0, p_2), (0, d_2)\}$.

The proof of Theorem 1 is given in the appendix.

Based on the above result, a moment should convince the reader that at the steady state, the general structure of the optimal schedule is as shown in Figure 1. In the latter, x_i represents the inventory/backlog axis of Product *i*. Now, assume that we start the cycle at the point A on the cycle, we then progress toward the point B by producing Product 1 at the demand rate along segment [A,B], where x_2 decreases until we reach point B. At this point, we increase the production rate to the maximum and continue producing Product 1. Between B and C, x_1 increases while x_2 keeps decreasing until we reach Point C, where we switch production to Product 2. During the setup of the machine for Product 2, both inventory levels decrease. Once the machine is ready to produce Product 2 (Point D), the production begins with the maximum allowable rate in order to eliminate backlog as soon as possible (since after the setup, we end up with a backlog for Product 2). Once Product 2 backlog is completely eliminated, we decrease the production rate to the demand rate so that along [E,F] the inventory of Product 1 decreases, that of Product 2 remains zero. When we reach Point F, the production rate of Product 2 is increased to the maximum and a certain inventory is built to hedge against future shortages brought about by setups and production of Product 1. At the point G, we setup the machine for Product 1. At the point H, we produce Product 1 at maximum production rate to eliminate backlog as soon as possible until we reach

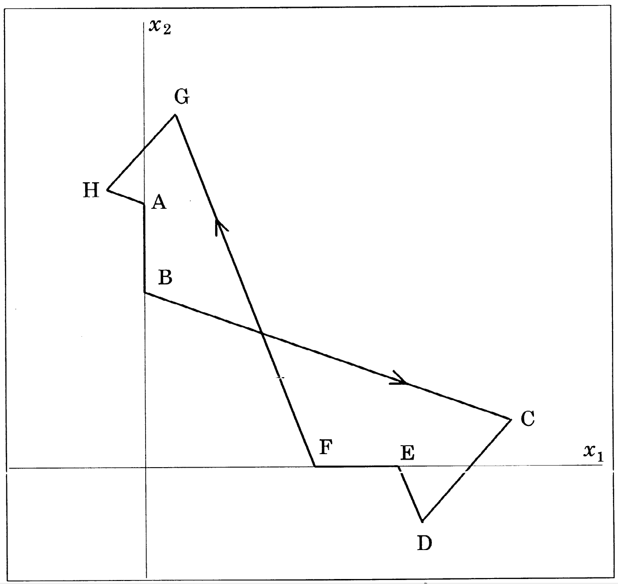

Figure 1. Illustration of the two-product common cycle schedule.

point A, where we start the cycle all over again. Based on the previous schedule, the problem can be stated as follows:

PROBLEM. *Given the above schedule, determine the optimal production times at the demand rate* τ_1*(Segment* [*A*,*B*] *in Figure 1*) *and* τ_2 (*Segment* [*E*,*F*]), *the optimal production times at the maximum rate* t_1 (*Segment* [*B*,*C*]) *and* t_2 (*Segment* [*F*,*G*]), *so as to minimize the average setup, inventory holding and backlog costs per unit time, provided that the demand is met over an infinite horizon.*

Notice that, the optimal shape of the cyclic schedule will be determined by the values of τ_1 and τ_2. For instance, if the optimal τ_1 and τ_2 are zero, we recover the case studied in Bai and Elhafsi (1996). In this case, F$\equiv E$ and A$\equiv$B in Figure 1.

Graphically, the optimization problem can be seen as one of locating and determining the shape of the cyclic schedule in the *x*-space so as to minimize the average setup, inventory holding and backlog costs per unit time. In the following, we formulate the problem mathematically and then derive the optimal solution in closed form. Notice here, that we do not consider idle times. The reason is as follows: the purpose of idling the machine (i.e. stopping the machine completely) during the production of a product is usually to stretch the cycle time and delay setup costs which may be high in some cases. In our case, this task is accomplished by producing the products at the demand rate which eliminates inventory and backlog costs for a product during its production at the demand rate along with delaying setup costs. In the case of fixed production rates, during idle times there is a certain inventory or backlog that is present for which a cost is incurred. Therefore idling cannot be optimal in our case.

3.1. Problem Formulation

Without loss of generality, let us start the cyclic schedule shown in Figure 1 at either Points A or E, and let us denote by *i* the index of the product we start with. According to Figure 1, the inventory of product *i* behaves as shown in Figure 2 where, S_i and s_i are the maximum inventory and backlog levels respectively attained within a cycle of product *i*. Note that s_i is non positive and S_i is non-negative. Let $Q_i = S_i - s_i$. This quantity is known as the replenishment quantity in the inventory theory literature. Obviously Q_i must be positive. Notice here that the production cycle is completely characterized when the quantities S_i, Q_i, and τ_i are determined. This, we do in the following.

Based on Figure 2, it is not difficult to see that the total inventory holding cost and the total backlog cost over a cycle are given as follows:

$$\textit{Inventory cost} = \frac{1}{2} h_i \frac{S_i^2}{d_i(1-\rho_i)};$$
$$\textit{Backlog cost} = \frac{1}{2} b_i \frac{s_i^2}{d_i(1-\rho_i)} = \frac{1}{2} b_i \frac{(S_i - Q_i)^2}{d_i(1-\rho_i)}.$$

The average setup, inventory holding and backlog cost per unit time of Product *i* is then given by:

$$F_i(S_i, Q_i) = \frac{k_i}{T} + \frac{1}{2}\frac{h_i}{T}\frac{S_i^2}{d_i(1-\rho_i)} + \frac{1}{2}\frac{b_i}{T}\frac{(S_i - Q_i)^2}{d_i(1-\rho_i)}.$$

The total average setup, inventory holding and backlog cost per unit time is given by:

$$F(S_1, S_2, Q_1, Q_2) = \sum_{i=1}^{2} \left(\frac{k_i}{T} + \frac{1}{2}\frac{1}{d_i(1-\rho_i)} \left(\frac{h_i S_i^2 + b_i (S_i - Q_i)^2}{T} \right) \right). \quad (1)$$

In the following, we show how *T* and Q_i are related to τ_i, the time spent producing Product *i* at the demand rate. First, it is not difficult to see that

$$t_i = \frac{Q_i}{d_i}\frac{\rho_i}{(1-\rho_i)}. \quad (2)$$

From the demand satisfaction constraint $Td_i = \tau_i d_i + t_i p_i$, we get

$$Q_i = d_i(1-\rho_i)(T - \tau_i). \quad (3)$$

But $T = \sum_{i=1}^{i=2}(t_i + \tau_i + \delta_i)$. Substituting for t_i and rearranging terms, we get

$$T = \frac{1}{(m-1)} \sum_{i=1}^{i=m} \frac{Q_i}{d_i} - \frac{\delta}{(m-1)} = \sum_{i=1}^{m} \frac{(1-\rho_i)}{(1-\rho)}\tau_i + \frac{\delta}{(1-\rho)}. \quad (4)$$

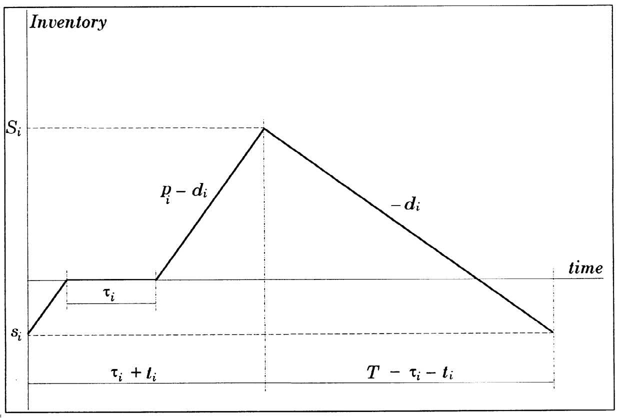

Figure 2. Inventory behavior of Product *i*.

Here *m*=2 (two products). Substituting *T* in the expression of Q_i, gives:

$$Q_i = q_i \left(\delta + \sum_{j=1}^{m} (1-\rho_j)\tau_j - (1-\rho)\tau_i \right) \Big/ \delta; \tag{5}$$

where $q_i = \delta d_i(1-\rho_i)/(1-\rho)$. q_i can be seen as the replenishment quantity, when the τ_i's are all zero. It is clear that the independent variables of the model are the S_i's and the τ_i's. Letting *S* and *t* be the vectors which components are the S_i's and the τ_i's respectively, the total average cost per unit time can be rewritten as follows:

$$F(S,\tau) = (m-1) \sum_{i=1}^{i=m} \left(k_i + \frac{1}{2} \frac{1}{d_i(1-\rho_i)} (h_i S_i^2 + b_i (S_i - Q_i)^2) \right) \Big/ \left(\sum_{i=1}^{i=m} \frac{Q_i}{d_i} - \delta \right). \tag{6}$$

The minimization problem can be stated as follows:

$(\mathcal{P})$ *Minimize* $F(S,\tau)$
Subject to :

$$Q_i = q_i \left(\delta + \sum_{j=1}^{m} (1-\rho_j)\tau_j - (1-\rho)\tau_i \right) \Big/ \delta, \quad i = 1,\ldots,m;$$
$$S_i \geqslant 0, \quad \tau_i \geqslant 0, \quad Q_i \geqslant 0, \qquad i = 1,\ldots,m.$$

Our solution approach consists of two steps. First, we obtain the optimal S_i's as a function of the τ_i's through the quantities Q_i. Second, we ignore the non-negativity constraints and solve for the τ_i's implicitly through the quantities Q_i. If the obtained solution is feasible (i.e., satisfies the non-negativity constraints), then it is optimal. If the non-negativity constraints are violated, then in the optimal solution of the constrained problem, either one or both non negativity constraints will be effective. Hence, we must distinguish the three possible constrained solutions. That is, $(\tau_1 = 0, \tau_2 > 0), (\tau_1 > 0, \tau_2 = 0)$, or $(\tau_1 = 0, \tau_2 = 0)$. A detailed procedure for obtaining the feasible optimal solution will be provided later in the section.

Setting the partial derivatives of $F(S, \tau)$ with respect to S_i to zero and rearranging terms gives:

$$S_i^* = Q_i b_i/(h_i + b_i), i = 1, \ldots, m. \tag{7}$$

The second partial derivative of $F(S, \tau)$ with respect to S_i is always positive. Therefore, $F(S, \tau)$ is strictly convex in S_i , and S_i^* given above is a global minimum for $F(S, \tau)$. Substituting S_i^* in $F(S, \tau)$ gives:

$$F(\tau) = \frac{K + A_1 Q_1^2 + A_2 Q_2^2}{Q_1/d_1 + Q_2/d_2 - \delta}. \tag{8}$$

PROPOSITION 1. *$F(\tau)$ is a strictly convex function of $\tau = (\tau_1, \tau_2)$.*

The proof is left for section 4, where we show that $F(\tau)$ is strictly convex for the multi-product case.

As a consequence of Proposition 1, $F(\tau)$ has a unique global minimum τ^* . In the following, we provide the optimal solution for the cases where $(\tau_1 \geqslant 0, \tau_2 \geqslant 0)$, $(\tau_1 = 0, \tau_2 \geqslant 0)$, $(\tau_1 \geqslant 0, \tau_2 = 0)$ and the case $(\tau_1 = 0, \tau_2 = 0)$.

Case $(\tau_1 \geqslant 0, \tau_2 \geqslant 0)$:
Taking the partial derivatives with respect to τ_1 and τ_2, setting to zero and rearranging terms gives the following system of two nonlinear equations:

$$\begin{cases} A_1(2\rho_2 - 1)Q_1^2 + A_2(1 - 2\rho_2)Q_2^2 + 2(\rho_2(d_1/d_2)A_1 \\ \quad +(1 - \rho_2)(d_2/d_1)A_2)Q_1 Q_2 - 2\delta(A_1 d_1 \rho_2 Q_1 \\ \quad +A_2 d_2(1 - \rho_2)Q_2) - K = 0 \\ A_2(2\rho_1 - 1)Q_2^2 + A_1(1 - 2\rho_1)Q_1^2 + 2(\rho_1(d_2/d_1) \\ \quad A_2 + (1 - \rho_1)(d_1/d_2)A_1)Q_1 Q_2 - 2\delta(A_2 d_2 \rho_1 Q_2 \\ \quad +A_1 d_1(1 - \rho_1)Q_1) - K = 0 \end{cases} \tag{9}$$

PROPOSITION 2. *The linear equation $A_1 d_1 Q_1 = A_2 d_2 Q_2$ is a solution of the above system.*

Proof. Substituting $Q_1 = (A_2/A_1)(d_2/d_1)Q_2$ in the first equation of (9) gives the second equation.

Solving (9) using Proposition 2, gives the following optimal values of Q_1 and Q_2 for the unconstrained problem (i.e., without the non negativity constraints):

$$Q_1^u = q_1 d_2 \gamma_2 \left(1 + \sqrt{1 + 2\frac{K}{\delta^2}\left(\frac{(1-\rho_1)}{\gamma_1 d_1} + \frac{(1-\rho_2)}{\gamma_2 d_2}\right)}\right) \Bigg/ \left(\frac{(1-\rho_2)}{(1-\rho)} d_1\gamma_1 + \frac{(1-\rho_1)}{(1-\rho)} d_2\gamma_2\right); \tag{10a}$$

$$Q_2^u = q_2 d_1 \gamma_1 \left(1 + \sqrt{1 + 2\frac{K}{\delta^2}\left(\frac{(1-\rho_1)}{\gamma_1 d_1} + \frac{(1-\rho_2)}{\gamma_2 d_2}\right)}\right) \Bigg/ \left(\frac{(1-\rho_2)}{(1-\rho)} d_1\gamma_1 + \frac{(1-\rho_1)}{(1-\rho)} d_2\gamma_2\right). \tag{10b}$$

The superscript *u* stands for unconstrained. Note that, at this stage, we have no guarantee that τ_1^u and τ_2^u satisfy the non negativity constraints.

Case ($\tau_1 = 0, \tau_2$ unconstrained):
A similar procedure as in the previous case leads to the following expression for Q_1^u and Q_2^u.

$$Q_1^u = (1-\rho_1)d_1\sqrt{(K + A_2(\delta d_2)^2)/(d_1^2(1-\rho_1)^2 A_1 + d_2^2\rho_1^2 A_2)} \tag{11a}$$

$$Q_2^u = d_2\delta + \rho_1 d_2\sqrt{(K + A_2(\delta d_2)^2)/(d_1^2(1-\rho_1)^2 A_1 + d_2^2\rho_1^2 A_2)}. \tag{11b}$$

Case (τ_1 unconstrained, $\tau_2 = 0$):
In a similar fashion, we obtain:

$$Q_1^u = d_1\delta + \rho_2 d_1\sqrt{(K + A_1(\delta d_1)^2)/(d_2^2(1-\rho_2)^2 A_2 + d_1^2\rho_2^2 A_1)} \tag{12a}$$

$$Q_2^u = (1-\rho_2)d_2\sqrt{(K + A_1(\delta d_1)^2)/(d_2^2(1-\rho_2)^2 A_2 + d_1^2\rho_2^2 A_1)}. \tag{12b}$$

Case ($\tau_1 = 0, \tau_2 = 0$):
This case is trivial. Indeed, setting τ_1 and τ_2 to zero in (5) gives:

$$Q_1 = q_1 \quad \text{and} \quad Q_2 = q_2.$$

PROPOSITION 3. *The following procedure gives the optimal solution of the constrained problem.*

PROCEDURE 1

Compute Q_1^u and Q_2^u using (10a) *and* (10b);
Compute τ_1^u and τ_2^u as follows

$$\tau_1^u = Q_1^u/d_2 - Q_1^u\rho_1/(1-\rho_1)d_1 - \delta; \tag{13a}$$

$$\tau_2^u = Q_1^u/d_1 - Q_2^u\rho_2/(1-\rho_2)d_2 - \delta; \tag{13b}$$

IF $\tau_1^u \geqslant 0$ *and* $\tau_1^u \geqslant 0$ *THEN*

$$\tau_1^* = \tau_1^u, \tau_2^* = \tau_2^u, Q_1^* = Q_1^u \text{ and } Q_2^* = Q_2^u;$$

STOP
ELSE
Compute Q_1^u *and* Q_2^u *using* (11a) *and* (11b);
Compute τ_1^u *and* τ_2^u *using* (13a) *and* (13b).
IF $\tau_2^u > 0$ *THEN*
Calculate $C2 = F(\tau_1^u, \tau_2^u)$ *using* (8)
ELSE
$C2 = \infty$
END
Compute Q_1^u *and* Q_2^u *using* (12a) *and* (12b);
Compute τ_1^u *and* τ_2^u *using* (13a) *and* (13b).
IF $\tau_1^u > 0$ *THEN*
Calculate $C1 = F(\tau_1^u, \tau_2^u)$ *using* (8)
ELSE
$C1 = \infty$
END
Let $Q_1^u = q_1$ *and* $Q_2^u = q_2 \Rightarrow \tau_1^u = 0$ *and* $\tau_2^u = 0$;
Calculate $C0 = F(\tau_1^u, \tau_1^u)$ *using (8)*
$((\tau_1^*, \tau_2^*), (Q_1^*, Q_2^*)) = \text{argmin}_{(\tau_1^u, \tau_2^u),(Q_1^u, Q_2^u)}\{C0, C1, C2\}$;
ENDIF

Proof. Expressing τ_1 and τ_2 as a function of Q_1^u and Q_2^u, we get:

$$\tau_1 = Q_2^u/d_2 - Q_1^u\rho_1/(1-\rho_1)d_1 - \delta;$$

$$\tau_2 = Q_1^u/d_1 - Q_2^u\rho_2/(1-\rho_2)d_2 - \delta;$$

now, if τ_1 and τ_2 are both non-negative, then the solution of the unconstrained optimization problem is feasible, and since $F(\tau)$ is convex in τ_1 and τ_2, it follows that the solution is optimal for the constrained problem as well. If the non-negativity constraints are violated, then the optimal solution is obtained by comparing the costs of the three possible cases (effective constraints) and picking the minimum. For the cases where only one of the non-negativity constraints is effective, if we obtain a nonfeasible solution (i.e., the other constraint must be effective too), we set its cost to infinity. This is because, when only one non-negativity constraint is effective, we set the variable for which the constraint is effective to zero and we solve an unconstrained optimization problem with respect to the other variable. Hence, we might obtain a nonfeasible solution. Which completes the proof. ∎

Once we obtain the optimal $\tau_1^*, \tau_2^*, Q_1^*$, and Q_2^* we go back and calculate the optimal S_i, s_i, t_i $(i = 1, 2)$ and *T*, which completely characterizes the optimal cyclic schedule.

The case where backlog is not allowed is obtained by letting b_i (the backlog cost rate) go to infinity. Which is basically replacing γ_i by h_i for $i = 1, 2$ in the above formulas. In this case, the s_i's become all zero.

In the following sub-section, we compare our results with other models found in the literature. We show that by allowing a certain production time at the demand rate, we achieve significant savings.

3.2. Numerical Examples

The examples used for the comparison are those proposed by Boctor (1982) to illustrate his algorithm for a two-product Basic Period schedule model. These two examples were then used by Lee and Surya (1989) to compare their new algorithm to Boctor's. Also, we use these two examples with the model of controllable production rates (no production at the demand rate) of Buzacott and Ozkarahan (1983). Notice that no backlog is allowed for the two examples.

EXAMPLE 1.

i	d_i	p_i	δ_i	k_i	h_i
1	20000	160000	0.0125	15	0.005
2	27000	162000	0.0250	25	0.004

EXAMPLE 2.

i	d_i	p_i	δ_i	k_i	h_i
1	3500	100000	0.5	2500	0.15
2	46500	100000	0.3	18500	0.005

The optimal solution for the examples is as follows:

Example 1:	$\tau_1^* = 0.4282$	$\tau_2^* = 0.4815$	$t_1^* = 0.0900$	$t_2^* = 0.1111$	$T^* = 1.1483$
Example 2:	$\tau_1^* = 6.0248$	$\tau_2^* = 0.0000$	$t_1^* = 0.2521$	$t_2^* = 6.1509$	$T^* = 13.2278$

Table 1 shows the computational results of this comparison. The upper part of each entry in Table 1 shows the average cost for each schedule, while the lower part shows how much savings are made with our schedule. For instance, in Example 2, our schedule average cost is 51.3% lower than that of Boctor's. Although we expected the Basic Period schedule to perform better than the Common Cycle schedule, the results show that by introducing a production a the demand rate, we

Table 1. Comparison savings for the two-product model

	Elhafsi and Bai	Boctor	Lee and Surya	Buzacott and Ozkarahan
Example 1	72.0	118.7	119.2	94.3
	–	64.9%	65.6%	31.0%
Example 2	3403.8	5149.3	4153.0	4144.1
	–	51.3%	22.0%	21.7%

achieve savings up to 66% on the average cost of the schedule. Notice that, even with a controllable production rate but without a production rate at the demand rate (Buzacott and Ozkarahan), we obtain lower average costs.

4. The Multi-Product Problem

In this section, we solve the *m*-product problem numerically. We adopt Zoutendijk's Algorithm (see Bazaraa and Shetty, 1979). To simplify the algorithm, we exploit the structure of the problem and implement the algorithm using closed form expressions. But first, let us give the formulation for the *m*-product problem.

4.1. Problem Formulation

It is not difficult to extend the two-product problem formulation to the *m*-product case. Using the expression of *T* given by (4) and substituting $Q_i (i = 1, 2)$ given by (5) in the objective function given by (8) gives the following formulation:

$$(\mathcal{P})\ \textit{Minimize}\ F(\tau) = \frac{1}{T}\left(K + \frac{1}{2}\sum_{i=1}^{m} H_i (T - \tau_i)^2\right)$$

Subject to :

$$T = T_0 + \sum_{i=1}^{i=m} \alpha_i \tau_i;$$
$$\tau_i \geqslant 0, i = 1, \ldots, m;$$

where,

$$\begin{aligned} H_i &= \gamma_i d_i (1 - \rho_i) \quad i = 1, \ldots, m; \\ \alpha_i &= (1 - \rho_i)/(1 - \rho) \quad i = 1, \ldots, m; \\ T_0 &= \delta/(1 - \rho). \end{aligned}$$

T_0 is the length of the cyclic schedule when the τ_i's are all zero. Notice that the above formulation is expressed as a function of the τ_i's only. It is not difficult to show that S_i , given by (7), is still optimal, which completely characterizes the optimal cyclic schedule. Before we proceed with the algorithm, we propose the following proposition.

PROPOSITION 4. *$F(\tau)$ is strictly convex in τ.*

Proof. To prove this result, we proceed in two steps. First, we establish the convexity of *F* in (T, τ) , then we establish the convexity of *F* in τ . Let $G(T, \tau) = F(\tau) = \frac{1}{T}\left(K + \frac{1}{2}\sum_{i=1}^{m} H_i(T-\tau_i)^2\right)$. The hessian matrix of $G(T, \tau)$ is an $(m+1, m+1)$ matrix given by the following:

$$H_G = \frac{1}{T}\begin{pmatrix} \frac{1}{T^2}\left(2K + \sum_j H_j\tau_j\right) & -\frac{H_1\tau_1}{T} & -\frac{H_2\tau_2}{T} & \cdots & -\frac{H_m\tau_m}{T} \\ -\frac{H_1\tau_1}{T} & H_1 & 0 & \cdots & 0 \\ -\frac{H_2\tau_2}{T} & 0 & H_2 & 0 & \vdots \\ \vdots & \vdots & & & 0 \\ -\frac{H_m\tau_m}{T} & 0 & \cdots & 0 & H_m \end{pmatrix}.$$

The minors of H_G are given as follows:

$$M_i = \frac{\prod_{k=1}^{i-1} H_k\left(2K + \sum_{j=1}^{m} H_j\tau_j\right)}{T^{i+2}}, \quad \text{for } i = 1, 2, ..., m+1,$$

where M_{m+1} represents the determinant of H_G.

Notice that the minors of H_G are strictly positive and therefore, H_G is positive definite and *G* is a strictly convex function in (T, τ). Now, let $\partial f/\partial x$ and $\partial^2 f/\partial x^2$ denote the gradient and the hessian of the function *f* with respect to the vector *x*, and let $W = (T, \tau)$. Then, after applying the chain rule, we obtain:

$$\frac{\partial F}{\partial \tau} = \frac{\partial G}{\partial \tau} = \left(\frac{\partial W}{\partial \tau}\right)^T \frac{\partial G}{\partial W}.$$

Here, $\left(\frac{\partial W}{\partial \tau}\right)$ is a $(m+1, m)$ and $\frac{\partial G}{\partial W}$ is a $(m+1, 1)$. The hessian of F, H_F, is obtained in the same fashion, using the chain rule.

$$H_F = \frac{\partial^2 F}{\partial \tau^2} = \left(\frac{\partial W}{\partial \tau}\right)^T \left(\frac{\partial^2 G}{\partial W^2}\right)\left(\frac{\partial W}{\partial \tau}\right).$$

Here, $(\frac{\partial^2 G}{\partial W^2})$ is equal to H_G. Notice that the term involving the second derivatives of *W* with respect to τ is equal to zero since *W* is a linear vector function of τ. Now, for every τ, $\tau^T H_F \tau = (\frac{\partial W}{\partial \tau}\tau)^T H_G(\frac{\partial W}{\partial \tau}\tau)$. Letting $\lambda = \frac{\partial W}{\partial \tau}\tau$, we have: $\tau^T H_F \tau = \lambda^T H_G \lambda > 0$. Therefore, the hessian of *F* is positive definite. The result follows immediately. ∎

The following proposition guarantees that the optimal τ_i's cannot be infinite.

PROPOSITION 5. *Problem $\mathcal{P}$ has always a finite optimal solution.*

Proof. Assume that the optimal solution of Problem $\mathcal{P}$ is not finite and label the products so that $\tau_1, \tau_2, \ldots, \tau_n$ are infinite and $\tau_{n+1}, \tau_{n+2}, \ldots, \tau_m$ are finite

($n \leqslant m$). Without loss of generality, let $\lambda = \tau_1 = \tau_2 = \cdots = \tau_n$. Substituting and rearranging terms in the expression of T gives $T = \lambda(T_0/\lambda + \sum_{i=1}^{i=n} \alpha_i + \sum_{i=n+1}^{i=m} \alpha_i(\tau_i/\lambda))$. It is clear that if $\lambda \to \infty$, then $\tau_1, \tau_2, \ldots, \tau_n$ will all go to infinity. Now, substituting in the objective function, we get:

$$F(\tau) = \left(K + \frac{1}{2}\lambda^2 \sum_{i=1}^{n} H_i(M_1 + M_2 - 1)^2 + \right.$$
$$\left. + \frac{1}{2}\lambda^2 \sum_{i=n+1}^{m} H_i(M_1 + M_2 - \tau_i/\lambda)^2 \right) \Big/ \lambda(M_1 + M_2).$$

Where, $M_1 = T_0/\lambda + \sum_{i=1}^{i=n} \alpha_i$ and $M_2 = \sum_{i=n+1}^{i=m} \alpha_i(\tau_i/\lambda)$. $\lim_{\lambda\to\infty} M_1 = \sum_{i=1}^{n} \alpha_i$ and $\lim_{\lambda\to\infty} M_2 = 0$. Hence, $\lim_{\tau\to\infty} F(\tau) = \infty$. Which is obviously not optimal since we can always choose a feasible finite solution with a finite objective value. The proof is complete. ■

Since the objective function is strictly convex and the solution is always finite, it follows that Problem $\mathcal{P}$ has a unique global solution.

4.2. Algorithm

First, we state Zoutendijk's algorithm (Bazaraa and Shetty, 1979) for minimizing a differentiable function *f* in the presence of linear constraints of the form $Ax \leqslant b$.

Initialization step: start with a feasible solution x_1. Let $k = 1$, and go to main step.

Main step:

1. given x_k, suppose that A^T and b^T are decomposed into (A_1^T, A_2^T) and (b_1^T, b_2^T) so that $A_1 x_k = b_1$ and $A_2 x_k < b_2$. Let d_k be an optimal solution to the following problem.

   $(P1)$ *minimize* $\nabla f(x_k)^T \mathrm{d}$
   subject to $A_1 \mathrm{d} \leqslant 0$
   $-1 \leqslant \mathrm{d}_j \leqslant 1$ for $j = 1, \ldots, m$;

   if $\nabla f(x_k)^T \mathrm{d}_k = 0$, stop; x_k is the optimal solution. Otherwise, go to Step 2.
2. let λ_k be an optimal solution to the following line search problem:

   $(P2)$ *minimize* $f(x_k + \lambda\, \mathrm{d}_k)$
   subject to $0 \leqslant \lambda \leqslant \lambda_{\max}$

   where

$$\lambda_{\max} = \begin{cases} \min\{\hat{b}_i/\hat{d}_i : \hat{d}_i > 0\} & \text{if not all } \hat{d}_i \text{ are negative} \\ \infty & \text{if all } \hat{d}_i \text{ are negative} \end{cases}$$

$$\hat{b} = b_2 - A_2 x_k$$
$$\hat{d} = A_2 d_k.$$

Let $x_{k+1} = x_k + \lambda_k \mathrm{d}_k$, identify the new set of tight constraints at x_{k+1} and update A_1 and A_2 accordingly. Let $k = k + 1$, and repeat step 1.

Notice that Problem P1 is a linear programming problem. Since in our case $A = -I_m$ and $b = 0$, where I_m is the $m \times m$ identity matrix. P1 can be solved by inspection as follows. Let $\nabla F_i(\tau)$ be the i^{th} component of the gradient of the objective function $F(\tau)$ in problem $(\mathcal{P})$.

$$\nabla F_i(\tau) = \left(\alpha_i \sum_{j=1}^{j=m} H_j(T - \tau_j) - H_i(T - \tau_i) - \alpha_i F(\tau) \right) \Big/ T. \tag{14}$$

Let $\mathcal{T}_k = \{i : \tau_i^k = 0\}$ be the set of indices for which τ_i is equal to zero at iteration k (binding constraints). Then P1 can be rewritten as follows:

$$\begin{array}{lll} \textit{minimize} & \displaystyle\sum_{i=1}^{i=m} \nabla F_i(\tau^k)\, \mathrm{d}_i & \\ \textit{subject to} & 0 \leqslant d_i \leqslant 1 & \text{for } i \in \mathcal{T}_k \\ & -1 \leqslant d_i \leqslant 1 & \text{for } i \notin \mathcal{T}_k \end{array}$$

Since for a linear programming problem the optimal solution is always at a vertex, The following procedure gives the optimal solution:

PROCEDURE 2
IF $i \in \mathcal{T}_k$ *THEN*
IF $\nabla F_i(\tau^k) > 0$ *THEN*

$d_i = 0$;

ELSE

$d_i = 1$;

END
ELSE
IF $\nabla F_i(\tau^k) > 0$ *THEN*

$d_i = -1$;

ELSE

$d_i = 1$;

END
END

The optimization problem P2 is a line search problem, which can be eliminated by calculating the optimal value of λ analytically. Differentiating $F(\tau_k + \lambda \mathrm{d}_k)$ with respect to λ and setting the result to zero gives the following quadratic equation in λ.

$$M_1 \sum_{i=1}^{i=m} H_i(M_1 - \mathrm{d}_i)^2\lambda^2 + 2M_2 \sum_{i=1}^{i=m} H_i(M_1 - \mathrm{d}_i)^2\lambda$$
$$+2M_2 \sum_{i=1}^{i=m} H_i(M_1 - \mathrm{d}_i)(M_2 - \tau_i^k) = 0;$$

where,

$$M_1 = \sum_{i=1}^{i=m} \alpha_i \, \mathrm{d}_i \quad \text{and} \quad M_2 = T_0 + \sum_{i=1}^{i=m} \alpha_i \tau_i^k.$$

The positive solution of the above quadratic is given by:

$$\lambda^* = -\frac{M_2}{M_1} + \sqrt{\frac{2K + \sum_{i=1}^{m} H_i \left[\frac{M_2}{M_1}(M_1 - d_i) + (M_2 - \tau_i^k)\right]^2}{\sum_{i=1}^{i=m} H_i(M_1 - d_i)^2}}. \tag{15}$$

$\hat{b}$ and $\hat{d}$ can be obtained as follows. Since $b = 0$ and $A_2 = -I_n$ (non-binding constraints), it follows that $\hat{b}_i = \tau_i^k$ and $\hat{\mathrm{d}}_i = -\mathrm{d}_i^k$ for $i \notin \mathcal{T}_k$. The algorithm can now be stated as follows:

Initialization step: start with a feasible solution τ^1. Let $k := 1$, and go to main step.

Main step:

1. Identify the set $\mathcal{T}_k$;
   Apply *PROCEDURE 2*: $\to \mathrm{d}^k$;
   IF $\nabla f(\tau^k)^T \mathrm{d}^k = 0$ THEN
      STOP;
      τ^k is the optimal solution.
   ELSE
      GOTO Step 2.
   END
2. $\hat{b}_i := \tau_i^k$ and $\hat{\mathrm{d}}_i := \mathrm{d}_i^k$ for $i \notin \mathcal{T}_k$.

$$\lambda_{\max} := \begin{cases} \min\{\hat{b}_i/\hat{\mathrm{d}}_i : \hat{\mathrm{d}}_i > 0\} & \text{if not all } \hat{d}_i \text{ are negative} \\ \infty & \text{if all } \hat{\mathrm{d}}_i \text{ are negative} \end{cases}$$

   $\lambda^k := \min(\lambda^*, \lambda_{\max})$;

   $\tau^{k+1} := \tau^k + \lambda^k \, \mathrm{d}^k$;

   $k := k + 1$;

   GOTO Step 1.

4.3. Numerical Example with Backlog Allowed

As an example, consider the ten-product problem proposed by Bomberger (1966), which we extend to the case where backlog is allowed. The backlog cost is taken to be 30 times that of inventory holding cost. This example has been used by Moon *et al*. (1991) (no backlog is allowed), where they normalized the data. The data for the example is the following.

i	d_i (unit/day)	p_i (unit/day)	δ_i (day)	k_i (\$)	h_i (\$/unit/day)	b_i (\$/unit/day)
1	1	15.2941	0.500	130	0.20896	6.2688
2	1	23.5294	0.750	200	0.03188	0.9564
3	1	100	0.500	110	0.02321	0.6963
4	1	18.75	0.125	10	0.01667	0.5001
5	1	47.5	0.250	30	0.01063	0.3189
6	1	80	0.125	20	0.00490	0.1470
7	1	400	1.000	310	0.00375	0.1125
8	1	300	0.250	50	0.00223	0.0669
9	1	150	0.125	5	0.00170	0.0510
10	1	300	0.125	5	0.00027	0.0081

The optimal solution is as follows:

i	τ_i^* (days)	t_i^* (days)	S_I^* (units)	s_i^* (units)
1	109.78	1.74	24.1	−0.80
2	0	5.80	126.4	−4.21
3	0	1.36	130.7	−4.36
4	0	7.28	125.0	−4.17
5	0	2.87	129.3	−4.31
6	0	1.71	130.4	−4.35
7	0	0.34	131.7	−4.39
8	0	0.45	131.6	−4.39
9	0	0.91	131.2	−4.37
10	0	0.45	131.6	−4.39

$$T^* = 136.46 \; \textit{days}$$
$$\textit{Average cost per day} = \$13.05.$$

In Moon *et al*. (1991), where backlog was not permitted, the average cost per day is \$13.26. Hence, by allowing backlog we obtain a slightly lower cost, 1.6% lower in this case.

5. Conclusion

In this paper, we have studied a version of the ELSP when the production rates are controllable during the production run of a product, and backlog is permitted. We derived the optimal solution of the two-product problem in closed form. For the multi-product problem, we proposed a very simplified version of Zoutendijk's algorithm which requires neither solving a Linear Programming sub-problem for finding a feasible direction, nor performing a line search procedure to determine the next improving solution. Comparison with previous results reported in the literature revealed that savings up to 66% can be obtained when controllable production rates are allowed. Also, comparison with a pure inventory controllable production rates model showed that slightly lower average cost can be obtained when backlog is allowed.

Appendix

Before we proceed with the proof of Theorem 1, we need to put the problem in an optimal control context. This, we do in the following preliminary section.

PRELIMINARY

In this section, some of the notation is new and some is introduced in the main part of the paper. The formulation is given for the two-product case.

Let $x_i(t)$ be the cumulative production surplus of Part Type i $(i = 1, 2)$ at time *t*; a positive value of $x_i(t)$ represents inventory while a negative value represents backlog. Let $u_i(t)$ be the controlled production rate of the machine producing Type *i* parts at time *t*. Let $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t))$ be the setup state vector of the machine at time *t*. Where, $\sigma_i(t)$, $\sigma_{ij}(t)$ $(j \neq i, i = 1, 2, j = 1, 2)$ are right continuous binary functions of *t*, such that $\sigma_i(t) = 1$ when the machine is ready to produce Type *i* parts and $\sigma_i(t) = 0$ otherwise; $\sigma_{ij}(t) = 1$ when the machine is undergoing a setup change from Part Type *j* to Part Type *i* and $\sigma_{ij}(t) = 0$ otherwise. Let *s*(*t*) be a nonnegative right continuous function of *t* which takes on the value δ_i at the beginning of each setup change to Part Type $i (i = 1, 2)$ and decreases with time. *s*(*t*) indicates whether a setup is completed or not. The state variable of the system is given by the vector $x(t) = (x_1(t), x_2(t))$. The variables $u(t) = (u_1(t), u_2(t))$ and $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t))$ are the control variables. We denote by (σ, u) the complete control vector. The problem can then be formulated mathematically as follows:

$$\textit{minimize } \overline{J}_\mu = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} g(x(s), \sigma(s))\, \mathrm{d}s$$

subject to:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = u_i(t) - \mathrm{d}_i; \tag{1}$$

$$\sigma_1(t) + \sigma_2(t) + \sigma_{1,2}(t) + \sigma_{2,1}(t) = 1; \tag{2}$$

if $\sigma_i(t^-) = 1$ and $\sigma_i(t) = 0$, then $s(t) = \delta_j$ and $\sigma_{ij}(t) = 1$; (3)

if $s(t^-) > 0$ and $\sigma_{ij}(t^-) = 1$, then $\dot{s}(t) = -1$ and $\sigma_{ij}(t) = 1$; (4)

if $s(t^-) = 0$ and $\sigma_{ij}(t^-) = 1$, then $\sigma_{ij}(t) = 0$ and $\dot{s}(t) = 0$ and $\sigma_j(t) = 1$; (5)

$u(t) \in \Omega(\sigma(t))$; (6)

for $i = 1, 2, \quad j = 1, 2, \quad i \neq j$.

$\dot{s}(t)$ denotes the time derivative of $s(t)$.

$$\Omega(\sigma(t)) = \{u(t) | 0 \leqslant u_i(t) \leqslant p_i \sigma_i(t), \quad i = 1, 2\}.$$

$$g(x, \sigma) = \sum_{i=1, j \neq i}^{i=2} (h_i x_i^+(t) + b_i x_i^-(t) + (k_i/\delta_i)\sigma_{ji}(t)).$$

$g(x, \sigma)$ represents the instantaneous total inventory, backlog and setup cost. Notice that $x_i^+(t) = \max\{x_i(t), 0\}$ represents inventory and $x_i^-(t) = \max\{-x_i(t), 0\}$ represent backlog. Here, the minimization is over all functions $\mu(x(t)) = (\sigma(t), u(t))$, such that $x(t)$, $\sigma(t)$ and $u(t)$ satisfy constraints (1)–(6).

PROOF OF THEOREM 1

The proof is based on the Hamilton–Jacobi–Bellman (HJB) equation. Throughout the proof, we assume that the optimal cost functional is differentiable in *x* and *t*. In fact, we will show later in this paper that the optimal state trajectory is continuous piecewise linear. Hence, the optimal cost will not depend explicitly on *t* and will be the sum of quadratics in *x* (since the cost rate is linear in *x*) and therefore differentiable in *x*. Let $J^*_{t_f}(x, t) = \min_\mu \int_0^{t_f} g(x(s), \sigma(s))\, \mathrm{d}s$ and $J^* = \min_\mu \overline{J}_\mu$. Here, we have an average cost formulation. The HJB equation (see Kushner and Dupuis (1992) for a formal derivation) is given by:

$$J^* = \min_\mu \{g(x, \sigma) + V_{x_1}(x, t)(u_1 - d_1) + V_{x_2}(x, t)(u_2 - d_2)\};$$

where $V(x, t) = \lim_{t_f \to \infty}(J^*_{t_f} - t_f J^*)$. When the machine is undergoing a setup change to a Part Type, there is no decision to make and (u_1^*, u_2^*) is forced to be equal to (0,0). Now, assume that we know the optimal setup state of the machine. Let $\sigma = (1, 0, 0, 0)$ be this setup state. That is, the machine can produce Part Type 1. In this case, the HJB equation can be rewritten as follows:

$$J^* = \min_{u \in \Omega(1,0)} \{g(x, \sigma) + V_{x_1}(x, t)(u_1 - d_1) + V_{x_2}(x, t)(u_2 - d_2)\}.$$

Now, notice that at each time instant *t*, if we knew $V(x, t)$, we would solve a linear programming problem for which u_1 and u_2 are the decision variables, $\partial V / \partial x_1$ and $\partial V / \partial x_2$ are the cost coefficients and $\Omega(1, 0)$ is the constraints set,

$\Omega(1,0) = \{(u_1,u_2)|0 \leqslant u_1 \leqslant p_1, u_2 = 0\}$, which is bounded and convex. We know that the solution of the above linear programming problem is always at a vertex of the constraint set $\Omega(1,0)$. That is, (u_1^*, u_2^*) is either equal to (0,0) (if $\partial V/\partial x_1 > 0$) or equal to $(p_1, 0)$ (if $\partial V/\partial x_1 < 0$). Furthermore, the solution is unique if the cost coefficient $\partial V/\partial x_1$ is nonzero. In the case $\partial V/\partial x_1 = 0$, the solution is not unique anymore since any solution (u_1^*, u_2^*) will not affect the objective function of the linear programming problem at time instant *t*. However, to keep the cost coefficient $\partial V/\partial x_1$ equal to zero at time instant $t + \delta t$, we should produce Part Type 1 at the demand rate d_1 so as to minimize the rate of increase of the cost function J^*. In this case (u_1^*, u_2^*) is equal to $(d_1, 0)$. A similar argument is used when the optimal setup state is $\sigma = (0,1,0,0)$. That is, the machine is set up for Part Type 2. ■

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